University of Nottingham
Department of Mechanical, Materials and Manufacturing Engineering

Computer Modelling Techniques

FE-01

STRESS ANALYSIS & MATHEMATICAL BACKGROUND

1.1 Uniaxial Loading

1.1.1 Uniaxial (one-dimensional) Stress and Strain Relationships

For a uniaxial loading situation, conventional (also called 'nominal' or 'engineering') strain is defined as the change in length per unit original (undeformed) length. Figure 1.1 shows a bar or strut of original length L_o and area A_o , and final length L_n and area A_n subjected to a uniaxial force F, for which the engineering (or nominal) strain can be expressed as follows:

$$\varepsilon_{engineering} = \frac{L_n - L_o}{L_o} = \frac{u}{L_o} \tag{1.1}$$

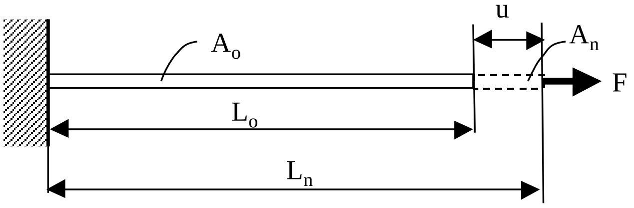


Figure 1.1: A uniaxial bar under tension

Stress is generally defined as the average force (F) per unit area (A). For a uniaxial loading situation, the stress can be simply defined as follows:

$$\sigma_{engineering} = \frac{F}{A_o} \tag{1.2}$$

This definition assumes that the stress is uniform over that particular area, but in reality stresses are seldom uniform over large areas. Therefore, it is more meaningful if this area is made very small, thus introducing the mathematical concept of "*stress at a point*" defined as follows:

$$\sigma = Lim_{\delta A \to 0} \left(\frac{\delta F}{\delta A} \right) \tag{1.3}$$

The concept of stress at a point is physically valid because a small area δA would carry a small amount of force δF.

1.1.2 Uniaxial stress-strain curves

Some basic ideas underlying the theory of elasticity and plasticity can be presented with reference to simple one-dimensional experimental tests on an elasto-plastic material, in which a test specimen in the shape of a cylindrical bar is subjected to a uniaxial tension F, as shown in Figure 1.2.

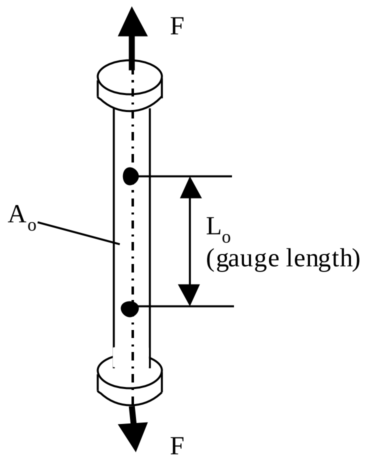


Figure 1.2: Uniaxial test specimen

Typical engineering (nominal) stress-nominal strain curves are shown in Figure 1.3(a) for low and medium carbon steels and Figure 1.3(b) for alloy steels and non-ferrous metals. These tests are usually performed under displacement control. If the test is conducted under load control, i.e. the load is gradually increased, the softening part of the stress-strain curve will not be featured.

The following observations can be made:

- The load rate or the strain rate influences the stress-strain curve.
- The stress linearly varies with strain until the yield point is reached.
- The stress value which separates the stress-strain curve into an elastic portion and a

plastic portion, is the *yield stress*, σ_{ys} .

- The maximum stress reached in the test is the *ultimate tensile stress (UTS)*,

- The strain reached when the specimen fails, *%A*, is the *percent elongation*, which is measure of the ductility of the material.

- In some materials, σ_{ys} can be easily identified from the stress strain curves, as shown in Figure 1.3(a) for low and medium carbon steels where two yield points can be identified, for which an *upper yield stress* and a *lower yield stress* are defined. However, often only a single value of the yield stress is used. Usually the lower yield stress is quoted.

- In some materials, such as alloy steels and non-ferrous metals, the transition from linear to non-linear stress is a gradual process with no clearly identifiable yield stress, as shown in Figure 3(b). In such materials, a stress quantity called the *proof stress* at a given plastic strain is defined. Typically, 0.1% and 0.2% proof stresses are used for engineering problems.

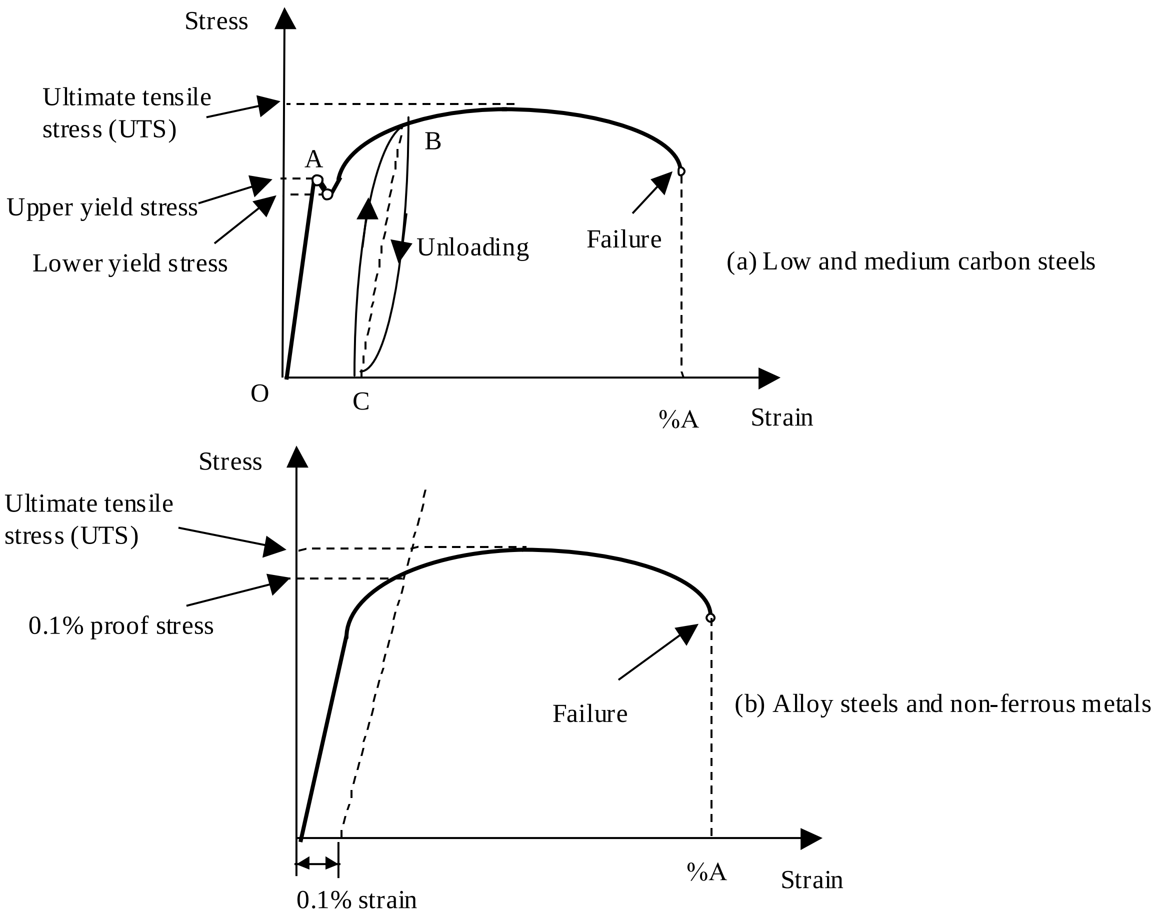


Figure 1.3: Typical uniaxial stress-strain curves

1.2 Three-Dimensional Stress and Strain

1.2.1 Multi-axial (3D) stress definitions

In a 3D Cartesian (x, y and z) axes system, there are six components of stress, as follows (see Figure 1.4)

- Three direct (tensile or compressive) stresses (σ_{xx} , σ_{yy} , σ_{zz}) caused by forces normal to the area

- Three shear stresses (σ_{xy} , σ_{xz} , σ_{yz}) caused by shear forces acting parallel to the area

The first subscript refers to the direction of the outward normal to the plane on which the stress acts, and the second subscript refers to the direction of the stress arrow

For simplicity, in most problems the first and second subscripts can be interchanged, i.e. $\sigma_{xy} = \sigma_{yx}$, $\sigma_{yz} = \sigma_{zy}$ and $\sigma_{xz} = \sigma_{zx}$ (*complementary shear stress*).

A "stress matrix" or a "stress vector", which contains all stress components, can be conveniently expressed as follows:

$$[\sigma] = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} \qquad (1.4)$$

Similarly, a "strain vector" can be defined as follows:

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{bmatrix} \qquad (1.5)$$

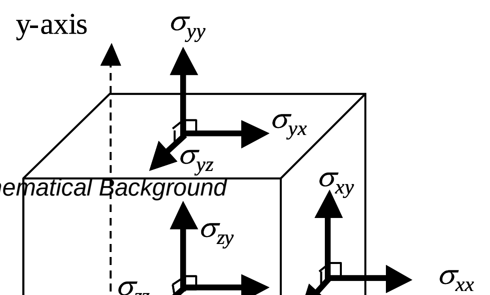


Figure 1.4: 3D Cartesian stresses

1.2.2 3D strain definitions

For 3D multi-axial problems, the uniaxial definition of strain as equal to $\Delta L/L$ is not applicable since there is no measurable 'length' in a 3D continuum.

3D strains can only be defined in terms of the displacements of the domain. The 3D direct (i.e. non-shear) strains are related to the displacement components as follows:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{1.6}$$

where u_x, u_y and u_z are the displacements (deformations) in the x,y and z directions, respectively

Note that the use of partial differentiation symbol (∂) in the above equations is intentional and indicates that the displacements (u_x, u_y and u_z) can be functions of the x, y and z coordinates. For example, it would be incorrect to express the strain ε_{xx} as (du_x/dx), since this would indicate that u_x is a function of only x (which is incorrect in a 3D domain), whereas ($\partial u_x/\partial x$) indicates that u_x is a function of x as well as other variables (in 3D problems, u_x is a function of x,y and z).

The 3D shear strains are defined as follows:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \qquad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \tag{1.7}$$

It should be noted that two definitions are often used for the shear strain, one with the 1/2 factor and one without. The shear strain definition with the 1/2 factor (often referred to as the *"mathematical shear strain"*) is mainly used for the convenience of use in tensor notations. The shear strain definition without the 1/2 factor is referred to as the *"engineering shear* *strain"*. Both definitions are valid, provided that the definition is followed throughout the derivation of other relationships involving strains.

The notation and sign convention used for strains are the same as those used for stresses, i.e. tensile strain (elongation) is positive while compressive strain is negative.

1.2.3 3D Stress-Strain Relationships (Hooke's Law)

Stress-strain relationships are often called *"Constitutive Equations"*. For isotropic linear elastic materials with thermal strain, the following 3D stress-strain equations (Hooke's law) can be used:

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] + \alpha(\Delta T) \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] + \alpha(\Delta T) \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] + \alpha(\Delta T) \\
\varepsilon_{xy} &= \frac{1}{2\mu}\sigma_{xy} \\
\varepsilon_{xz} &= \frac{1}{2\mu}\sigma_{xz} \\
\varepsilon_{yz} &= \frac{1}{2\mu}\sigma_{yz}
\end{aligned}
\tag{1.8}
$$

where

E = Young's modulus (units: N m^{-2})
ν = Poisson's ratio (no units)
μ = Shear modulus (units: N m^{-2})
α = Coefficient of thermal expansion (units: per ^{o}C)
ΔT = Temperature change from a reference value (units: ^{o}C)

The shear modulus μ is defined as follows:

$$\mu = \frac{E}{2(1+\nu)} \tag{1.9}$$

Hooke's law is usually written with strains placed on the left hand side of the equations, i.e. strains are expressed as a function of the stresses. However, in computational mechanics formulations, it is often more convenient to place stresses on the left hand side of the equations, i.e. stresses expressed as functions of strains. This can be easily achieved by algebraic manipulation of Hooke's law such that the stresses are on the left hand side, resulting in the following matrix expression, often referred to as the *"material constitutive equation"*:

$$[\sigma] = [D]\,[\varepsilon] \tag{1.10}$$

where [D] is called the *"elastic property matrix"*.

1.3 Energy Approaches

1.3.1 Stable and Unstable Problems

Static equilibrium can be stable, unstable or neutral. For example, a ball inside a concave surface is in stable equilibrium because, if displaced sideways, it will try to return to its rest position, i.e. the system reaches a position such that its potential energy is minimum. On the other hand, a ball on a convex surface will not return to its position if displaced sideways and the potential energy is maximum. Neutral equilibrium is achieved when the ball is placed on a frictionless flat surface, i.e. if displaced sideways it will stay in the new position and will not return to its old position. Here, the potential energy is unchanged with the displacement. Figure 1.5 summarises the three types of static equilibrium.

In linear FE or structural analysis, the objective of the analysis is always to find a solution in which equilibrium is 'stable'. The *principle of minimum total potential energy* (*T.P.E.*) is often used in FE formulations by expressing the problem in terms of the independent variables (usually displacements) and then minimising the *T.P.E.* with respect to the displacements.

1.3.2 Strain Energy

The work done by external forces on the body is stored in the form of strain energy. This strain energy is released upon the removal of the applied loads and the body returns to its un-deformed state.

For linear elastic behaviour, an expression for the elastic strain energy can be derived as follows:

$$U = \frac{1}{2}[\sigma][\varepsilon] \times volume \tag{1.11}$$

If the material behaviour is non-linear, a more general expression can be written as follows:

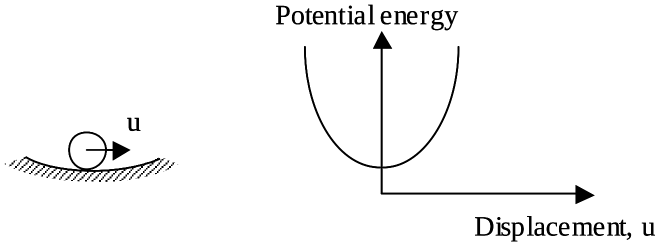
$$U = \int_v \int_\varepsilon \sigma \, d\varepsilon \, dV \tag{1.12}$$

1.3.3 Work Done by External Forces

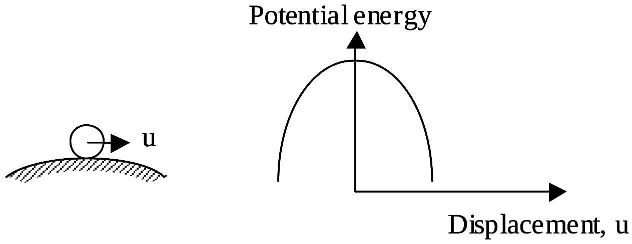
Another form of potential energy arises from the work done by the external forces that cause deformation of the body. This energy can be written as follows:

$$W = \sum_i F_i u_i \tag{1.13}$$

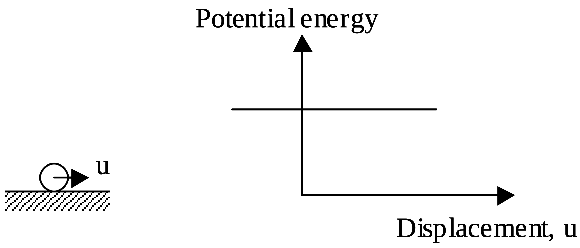
where i is any point where the force F_i causes a displacement u_i. More general expressions can be derived for distributed loads or body forces such as centrifugal or gravitational loads.



(a) Stable equilibrium



(b) Unstable equilibrium



(c) Neutral equilibrium

Figure 1.5: Static equilibrium states

1.3.4 The Principle of Minimum Total Potential Energy (T.P.E)

The total potential energy (*T.P.E.*) can be expressed as the difference between the strain energy and the work done by the external forces, as follows:

$$T.P.E. = U - W \qquad (1.14)$$

The *principle of minimum total potential energy* states that when the body is in equilibrium, the value of the *T.P.E.* must be ‘stationary’ with respect to the variables of the problem. The equilibrium is stable if the *T.P.E.* is minimum (and unstable if the *T.P.E.* is maximum).

In most FE formulations, the displacement, *u*, is chosen as the unknown (i.e. independent) variable of the problem, and the principle of minimum *T.P.E.* requires that the *T.P.E.* is stationary with respect to the displacements (not forces or stresses), i.e.

$$\frac{\partial(T.P.E.)}{\partial u} = 0 \qquad (1.15)$$

In FE formulations, it can be shown that the stationary *T.P.E.* is minimum. This minimisation of the *T.P.E.* can be done for each element in a finite element model, assuming that the interfaces between the elements make no contribution to the *T.P.E.*

1.4 Some Mathematical Background on Matrices

Using Matrices to represent equations

Matrices are often used to simplify writing length algebraic equations. For example, the following three equations:

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 &= b_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 &= b_2 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 &= b_3 \end{aligned} \tag{1.16}$$

can be expressed as matrices as follows:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{1.17}$$

or, in a more concise form:

$$[A]\,[x] = [b] \tag{1.18}$$

where, in this example, $[A]$ is a 3 x 3 matrix, $[x]$ and $[b]$ are 3 x 1 matrices.

In general, equation (1.18) can be used to represent any number of equations. If N is the total number of equations, then $[A]$ is a N x N matrix, $[x]$ and $[b]$ are N x 1 matrices.

Note that matrices such as $[x]$ and $[b]$ with just one column are sometimes called "*vectors*".

Matrix Multiplication

By observation of equations (1.16) and (1.17), the matrix multiplication rules can be easily learnt. In general, if two matrices $[A]$ and $[B]$ are multiplied, then the number of columns of $[A]$ must be the same as the number of rows of $[B]$, i.e. if $[A]$ is a (m x n) matrix, and $[B]$ is a (p x q) matrix, then n must be equal to p. The resulting matrix $[C]$ is a (m x q) matrix.

$$[A]^{(m\times n)} \times [B]^{(p\times q)} = [C]^{(m\times q)} \quad .... (n \text{ must be equal to } p) \tag{1.19}$$

Note that, in general, $[A]$ x $[B]$ is not equal to $[B]$ x $[A]$

Transpose of a Matrix

If the rows and columns of a matrix $[A]$ are interchanged, the resulting matrix is called the transpose of $[A]$ or $[A]^T$. If the elements of $[A]$ are written as a_{ij}, then the elements of $[A]^T$ are a_{ji}.

The following example shows a matrix $[A]$ and its transpose $[A]^T$:

$$[A] = \begin{bmatrix} 4 & 2 & 6 & 9 \\ 3 & 7 & 8 & 2 \\ 17 & 5 & 5 & 11 \\ 22 & 7 & 8 & 1 \end{bmatrix} ; \; [A]^T = \begin{bmatrix} 4 & 3 & 17 & 22 \\ 2 & 7 & 5 & 7 \\ 6 & 8 & 5 & 8 \\ 9 & 2 & 11 & 1 \end{bmatrix} \tag{1.20}$$

The following relationships are useful:

$$\begin{aligned} ([A] \times [B])^T &= [B]^T \times [A]^T \\ \left([A]^T\right)^T &= [A] \end{aligned} \tag{1.21}$$

Symmetric Matrix

A square matrix (number of rows equal to the number of columns) is called "*symmetric*" if $[A]^T = [A]$, i.e. $a_{ij} = a_{ji}$. This means that matrix coefficients above the diagonal of the matrix are "mirror images" of those below the diagonal. For example, the following square (4x4) matrix is symmetric:

$$[A] = \begin{bmatrix} 4 & 2 & 6 & 9 \\ 2 & 7 & 8 & 2 \\ 6 & 8 & 5 & 7 \\ 9 & 2 & 7 & 1 \end{bmatrix} \tag{1.22}$$

In FE formulations, the stiffness matrices are symmetrical, and it is important to exploit this symmetry to economise on the storage requirements of large matrices.

Inverse of a Matrix

A "*unit matrix*", $[I]$, is a square matrix in which all the coefficients of the principal diagonal are equal to 1, while all other coefficients are zero, as follows:

$$[I] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.23}$$

If for a given matrix $[A]$ there exists a matrix $[B]$ such that $[A][B] = [I]$, where $[I]$ is a *"unit matrix"*, then $[B]$ is called the "inverse" of $[A]$ and is denoted by $[A]^{-1}$ as follows:

$$[A] \times \left\lfloor A^{-1} \right\rfloor = [I] \tag{1.24}$$

Therefore, to solve a system of linear algebraic equations, such as that shown in equation (1.18), both sides of the equation can be multiplied by $[A]^{-1}$ to give:

$$[x] = \left\lfloor A^{-1} \right\rfloor [b] \tag{1.25}$$

However, for large matrices, such as those associated with FE analysis, it is important to use efficient numerical methods for solving large systems of algebraic equations. Inverting a large matrix requires a substantial number of mathematical operations, e.g. of the order of N^4 where N is the number of equations.

In practice, the direct computation of the inverse of $[A]$ is avoided, because it is very "expensive" (i.e. requires a substantial amount of computational time). Instead, special equation solving methods such as "*Gaussian Elimination*" or iterative "*Gauss-Seidel*" techniques are used.

An Example of Using Matrices in Equations

To demonstrate the advantages of using matrices in FE formulations, consider a one-dimensional (uniaxial) problem where the strain energy stored in the body, per unit volume, is given by:

$$U = \frac{1}{2}\,\sigma_{xx}\ \varepsilon_{xx} \qquad (1.26)$$

This expression can be generalised for two-dimensional problems, as follows:

$$U = \frac{1}{2}\left(\sigma_{xx}\ \varepsilon_{xx} + \sigma_{yy}\ \varepsilon_{yy} + \sigma_{xy}\ \varepsilon_{xy}\right) \qquad (1.27)$$

Similarly, for three-dimensional problems:

$$U = \frac{1}{2}\left(\sigma_{xx}\ \varepsilon_{xx} + \sigma_{yy}\ \varepsilon_{yy} + \sigma_{zz}\ \varepsilon_{zz} + \sigma_{xy}\ \varepsilon_{xy} + \sigma_{xz}\ \varepsilon_{xz} + \sigma_{yz}\ \varepsilon_{yz}\right) \qquad (1.28)$$

Rather than dealing with three equations, it is much more convenient to express all the above equations in a single equation applicable to any dimensionality. Using matrices, all 3 equations can be combined as follows:

$$U = \frac{1}{2}\,[\sigma]^T\,[\varepsilon] \qquad (29)$$

where $[\sigma]$ and $[\varepsilon]$ are the stress and strain vectors, respectively.

Note that $[\sigma]^T$, i.e. the transpose of the stress matrix, is used in equation (1.29) in order to satisfy the matrix multiplication rules. Since $[\sigma]^T$ is a 1 x 6 matrix and $[\varepsilon]$ is a 6 x 1 matrix, their product is a 1 x 1 matrix, i.e. a matrix with only one coefficient.

An Alternative Tensor Notation

An alternative notation, called the "*tensor*" notation, is also widely used in computational mechanics formulations. This notation is based on using subscripts such as i, j, and k as follows:

$$U = \frac{1}{2}\,\sigma_{ij}\ \varepsilon_{ij} \qquad (1.30)$$

where the subscripts i and j take the values 1, 2 and 3 corresponding to the Cartesian directions x, y and z, respectively.

Both matrix and tensor notations are widely used in computational mechanics formulations. **Only matrix expressions will be used hereafter.**